Reprinted from Barbara Moses, ed., *Algebraic Thinking, Grades K–12: Readings from NCTM's School-Based Journals and Other Publications* (Reston, Va.: National Council of Teachers of Mathematics, 2000), pp. 112–19. Originally appeared in *Teaching Children Mathematics* 3 (February 1997): 282–89.

Experiences with Patterning

Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips

Over the past decade we have learned that children are capable of mathematical insights and mathematical invention that exceed our expectations. We have also learned that we, as teachers, contribute to—or suppress—this insight and inventiveness in our students by the choices we make. We choose the mathematical tasks, the questions, and the expectations for how students are to interact with those tasks and with one another around those tasks. The question of the expectations we knowingly or unknowingly set for our students is nowhere more crucial than in the gatekeeper area called algebra. In this article we will share an example of how algebraic thinking and reasoning might be extended over grades K–6. We hope to stimulate readers to think with us about how we can search for ways to foster algebraic thinking and reasoning by the questions we regularly ask our students.

Joan Ferrini-Mundy, jfmundy@christa.unh.edu, is affiliated with the University of New Hampshire, Durham, NH 03824. She is interested in changes in educational practice in grades K–16. Glenda Lappan, glappan@math.msu.edu, and Elizabeth Phillips, ephillip@math.msu.edu, are codirectors of the Connected Mathematics Project at Michigan State University, East Lansing, MI 48824-1050. They share an interest in the teaching and learning of mathematics along with the professional development of teachers.

Although we have no easy answer as to what constitutes algebra or algebraic thinking and reasoning, we can be guided by the view of algebra that emerges from an examination of the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989). Standard 13 in the K–4 section, titled Patterns and Relationships, and Standards 8 and 9 in the 5–8 section, titled Patterns and Functions, and Algebra, respectively, suggest that the study of patterns is a productive way of developing algebraic reasoning in the elementary grades. Current curriculum reform efforts and research in learning contend that observations of patterns and relationships lie at the heart of acquiring deep understanding in many areas of mathematics—algebra and function in particular (Steen 1988). When students are presented with interesting problems in context, they observe patterns and relationships: they conjecture, test, discuss, verbalize, generalize, and represent these patterns and relationships. Generalizing and representing patterns are reflected in the example that follows.

An Example for Developing Algebraic Thinking

The following situation was adapted from the NCTM's Algebra Working Group (1995). This situation offers algebraic explorations in grade K–8 or beyond.

> Tat Ming is designing square swimming pools. Each pool has a square center that is the area of the water. Tat Ming uses blue tiles to represent the

water. Around each pool there is a border of white tiles. Here are pictures of the three smallest square pools that he can design with blue tiles for the interior and white tiles for the border. (See fig. 1.)

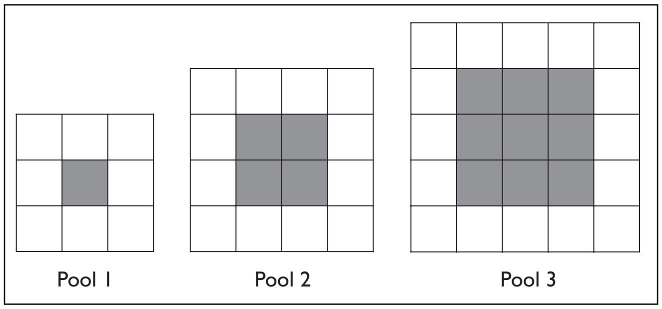

Fig. 1. Swimming pools with borders

What patterns, conjectures, and questions will children find as they explore this situation? Where is the algebra? Let us think about tasks and questions that would fit the various grade levels K–6 The intent of each question is to prompt students to look for patterns among the variables, make conjectures, provide reasons for their conjectures, and represent their patterns and reasoning. The questions and grade levels are only suggestions. Children will generate their own ideas and will pursue their own interests, using this situation as a starting point.

Grades K–2

Kindergarten children will be interested in the colors and in counting the tiles. Two kinds of tiles are used, and the number of each is not necessarily the same. Let us focus on beginning relationships between the numbers of blue and white tiles.

- For each square pool, sort the tiles into blue tiles for the water and white tiles for the border.
- Count how many tiles are in each pile.
- Are there more blue tiles than white tiles?

Next we take the problem a bit further by looking at the pattern in the blue-tile squares alone.

Here are pictures of the three smallest squares that Tat Ming has designed for the water. (See fig. 2.)

- Build each of the three blue squares. How many blue tiles are in each square?
- Build the next-biggest square that you can make out of the blue tiles. Then build the next. Count the squares in each.
- What patterns do you see?
- What is a square?

Here we can return to the original setting and look at the patterns in the two kinds of tiles in figure 1.

- Build the three pools using blue and white tiles to show the water and the border tiles. Record the information in a table. (See table 1.)
- How many tiles will be in the next-largest pool? Check your answer by building the square.
- Describe your methods for counting the different tiles.
- What patterns do you see?

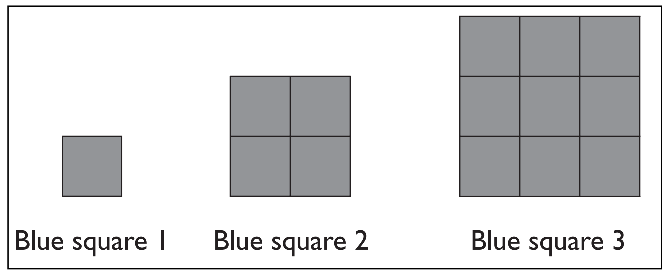

Fig. 2. The pool without borders

Table 1
Organizing the Data

Pool Number	Number of Blue Tiles	Number of White Tiles	Total Number of Blue and White Tiles

Looking into the Classroom for Grades K–2

The beginning questions engage students in sorting and counting the blue and white tiles. This activity helps them to look at the relationship between the numbers of blue and of white tiles. Students might observe that the first three pools have more white tiles than blue tiles. The teacher may ask if this situation is always true and encourage the students to

build the next-larger pool. This pool contains more blue tiles than white tiles. At this level, students represent their thinking and conjectures with objects that are concrete in nature.

In a first-grade class, some students focus on the blue tiles and what it means to be square. They may notice that there are as many rows as columns in the figures. Some may find convenient ways to step-count to find the number of tiles: “two, four” or “three, six, nine.” Some students may begin to guess the number of tiles in the next-larger blue square. The teacher can follow these observations by asking students how many tiles are on each edge of the blue square. Students are beginning to see a connection between the number of tiles needed to build a square and the length of its edge. Their observations are made and checked using the tiles.

In a second-grade class, students begin to organize their data into a table. They use newly developed computational skills to find ways to multiply and add.

Grace, a second grader, at first attended only to the overall size of the squares, not to the differences in color. In exploring this situation, Grace worked with the teacher to reach a definition of a square. This transcription illustrates the tentativeness of the student’s concept of square and her need to work with concrete materials to help herself think about the concept. Working in such a relatively open-ended setting can often reveal unexpected student thinking about concepts that we assume students understand thoroughly.

T: Why do you call it a square? What’s a square to you?

G: A block.

T: If it were longer down like this, would it still be a square?

G: No, it would turn into a rectangle.

T: So what makes it a square?

G: That it’s not as far down as a rectangle.

T: Is there anything else about the sides? How long is this side?

G: Three squares.

T: How long is this side?

G: Three squares.

T: So what is a square?

G: Can I try something? I’m putting out three to see if I can scramble them around and make a square. [Grace is working with three unit squares and trying to build a square. Notice that the teacher was trying to draw Grace’s attention to the equality of the sides. But, not unexpectedly, she became interested instead in the number 3 and its relationship to the square.]

T: A square out of three? [Grace notices that she will need four squares to make a square and builds it.]

T: How can you be sure it’s a square?

G: You can, because all the sides are the same length.

She was also very interested in counting the number of small tiles in each pool. She counted by threes for pool 1, by fours for pool 2, by fives for pool 3, and was able to predict the total number of squares in the fourth pool by intuitively applying the associative property of addition to compute 6 $\times$ 6 as follows:

$$\begin{aligned} 6 \times 6 &= [(6 + 6) + (6 + 6)] + (2 \times 6) \\ &= (12 + 12) + 12 \\ &= 24 + 12 = 36 \end{aligned}$$

Grace was quite intrigued by the prediction and computed the number of tiles in the seventh pool.

$$\begin{aligned} 7 \times 7 &= [(7 + 7) + (7 + 7) + (7 + 7)] + 7 \\ &= (14 + 14 + 14) + 7 \end{aligned}$$

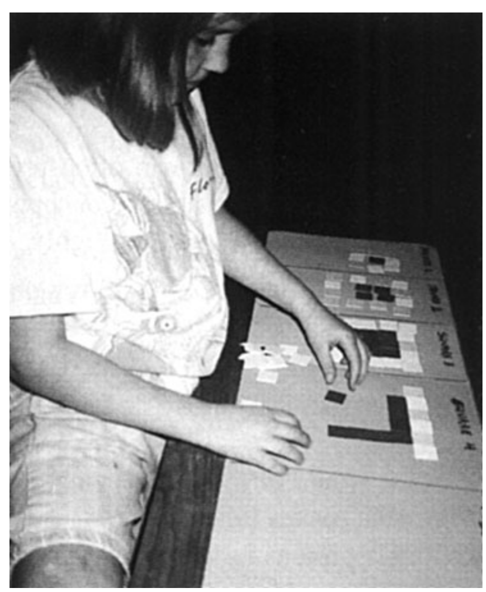

Fig. 3. Grace builds a square.

Grace was searching for a way to find the total number of squares in a pool. This example illustrates an aspect of algebra that involves developing and generalizing algorithms.

She also filled out the table (see fig. 3). She noticed that the four corners would always be present, which seemed to help her in figuring out how to count the border. She physically moved the corner squares away.

Grades 3–4

Using the same basic situation, we can begin to ask questions that encourage students to reason about the patterns in the number of blue and white tiles for a *given* pool and to reason about the number of border tiles *given the number of blue tiles* and the number of blue tiles *given the number of border tiles.* (See fig 1.)

- Build the first 3 pools and record the data in a table. (See table 1.)
- Continue the table for the next 2 squares. How do you know your answers are correct?
- If there are 32 white tiles in the border, how many blue tiles are there? Explain how you got your answer.
- If there are 36 blue tiles, how many white tiles are there? Explain how you got your answer.
- Can you make a square with 49 blue tiles? Explain why or why not.
- Can you make a square with 12 blue tiles? Explain why or why not.

By grade 4, students are learning to make comparisons by looking at the fraction or proportion that a part is of the whole. We can use a version of our problem to give students a new context for using fractions by drawing on patterns. (See fig. 1)

- In each of the first three square pools, decide what fraction of the square's area is blue for the water and what fraction is white for the border.
- What patterns do you see?
- What fractions will occur in the next two rows of the table? How do you know that your answers are correct? (See table 2.)

Looking into the Classroom for Grades 3–4

In grade 3, the relationship between the number of blue tiles and that of white tiles comes back into play. The teacher can foster the habit of looking for patterns and relationships between the variables by asking, "As the pools get larger and larger, what happens to the number of white tiles and the number of blue tiles?" Students may observe that both numbers are increasing but that for the first three squares, more white tiles are found than blue tiles.

Starting with the fifth square, more blue tiles are seen. These observations lead to some beginning insights into different kinds of growth patterns. As the students look for patterns in the table, some may observe that to get the number of white tiles for the next square pool, you always add four.

Teachers' questions can foster the habit of looking for patterns and relationships.

Ryan, a fourth-grade student, got interested in the patterns he could see in the table and noted, while looking at the squares rather than the table, that "it goes up by fours" in the border (white tiles) column. The teacher asked him why, and he provided a nice geometric explanation.

Table 2
Looking for Fraction Patterns

Pool Number	Total Number of Blue and White Tiles	Fraction of Blue Tiles for the Water	Fraction of White Tiles for the Border

The following exchange with Ryan illustrates how he used the physical representation to account for the pattern of “going up by fours” that he noted in the border column of the table. At first he associates the four corners with this increment of four. He then finds a more satisfying explanation.

T: Why does it go up by fours?

R: I think it goes up by fours because there’s four corners in each. So if you take them out, there’s one square on each border [looking at first square], so this would be four, this would be eight .... [He decides that this explanation is not adequate.]

T: Why does it go up by four? When it goes by four, from the first to the second....

R: Wait! Wait! Wait! I get it now. You see this is four [points to the four white squares bordering the blue in the first pool]. Then this goes up another four [looking at the second pool]. This has two [referring to the side of the blue square in the second pool].

T: Why does it go up by four?

R: This is one [referring to the blue square in pool 1]. It only has one white square on each side. This has two [referring to the side of the blue square in pool 2]. So it multiplies, I mean goes up by four because you add a white one to each [new] side. (See fig. 4.)

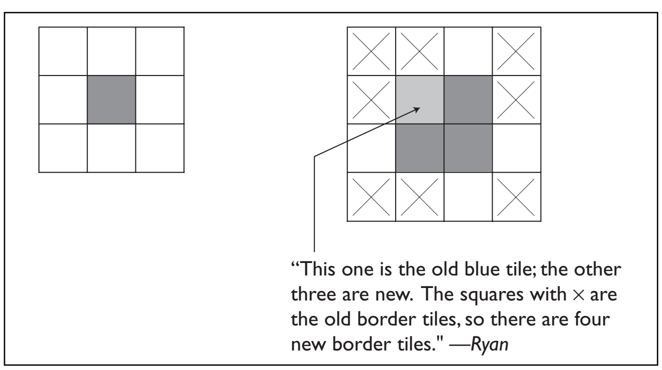

Fig. 4. Ryan explains that the number of border tiles increases by 4 each time.

Going from one to the next, you pick up four new border squares because you have added four new “outside” edges.

Ryan was intrigued by trying to make the connections between the manipulatives and the table and by trying to justify what he could see in the patterns of the numbers with what he could do with the squares. He had an algebraic definition for squares—a number is square if another number, when multiplied by itself, equals the number—as well as a tentative geometric interpretation. In the following sequence of dialogue with Ryan, we see him making connections between a previous definition of square number and the physical setting.

T: Tell me about the sixth pool.

R: Pool 6 would have 8 across, so that would be 8 times 8, or 64. The total would be 64 [reasoning from the manipulatives].

T: How did you figure that out?

R: There would be 8 across, 8 going down, and 8 times 8 would be 64. The number in the border would be 28.

T: How did you figure that out?

R: ‘Cause it goes with the patterns. [Note the ease with which he moves from the physical materials into the table.] Then I’m going to figure this out: 64 minus 28—whatever is left over would be the number of blue [again, he is reasoning from the table and using computation to solve the problem]. Okay—36.

T: Are you pretty sure that’s right?

R: [He checks by adding 36 and 28.]

T: Would it be a better check to make this square, or are you pretty sure? What if you were going to build it? If you made a square with each side 6, would you get 36 squares in it?

R: Yeah. If you use 6 … because 6 times 6 is 36.

T: And that’s what it means to make 6 squared. Have you ever heard of 6 squared—6×6?

Notice in the following dialogue the tentativeness of Ryan’s geometric definition of a square number and how continued exploration in this setting enables him to become more consistent in using his geometric understanding of an algebraic equation.

R: No, but I’ve heard of square numbers.

T: Is 36 a square number?

R: Yes.

T: Tell me why.

R: Because you can make a square with 36 squares.

T: Show me on the chart which are square numbers.

R: [Ryan looks at the numbers in the border column to see if they are square numbers. He tries to build a square of area 8 and cannot.]

T: Where are the square numbers in the table? You told me a square number was one you could make into a square.

R: Right. So all the squares: 9, 16, 25.... Those were the squares!

Ryan's teacher also asked questions about fractions. Even though the fraction questions were new to Ryan, he quickly made a table after the teacher started it. The table shows the fraction of blue squares to the total squares in the first few figures. (See table 3.) When asked if the number of blue tiles would be half the total tiles, he said, "Well, it won't be half blue until the border and the number of blues are the same." He looked at the table and noted that it was close in the fifth square (25 out of 49) but not equal. Ryan is beginning to see patterns in equivalent fractions. (See fig. 5.)

Ryan also is beginning to see that the linear pattern of the white tiles is overtaken by the quadratic pattern of the blue tiles even though he does not know the names for these patterns of growth. Both Grace and Ryan used a lot of computation in the process of looking for patterns. Grace added and counted in multiples, and Ryan multiplied and subtracted.

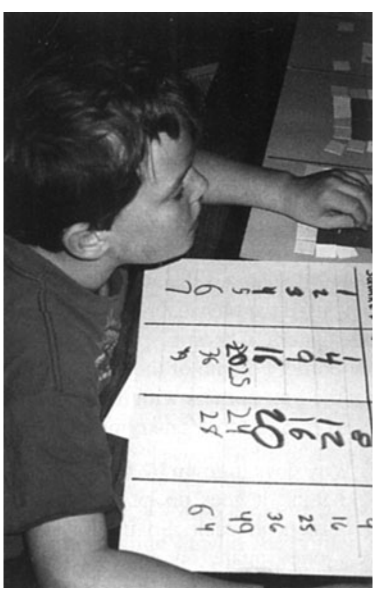

Fig. 5. Ryan completes his table.

Table 3
Ryan's Fraction Table

1st	1/9
2nd	1/4 (4 out of 16)
3rd	9/25
4th	16/36

Grades 5–6

In grade 5, we can use new ways to represent the relationships between the number of tiles of each color and the number of the square pools. We can begin to make the emphasis on function more explicit. (See fig. 1.)

- Make a table showing the numbers of blue tiles for water and white tiles for the border for the first six square pools.
- What are the variables in the problem? How are they related? How can you describe this relationship in words?
- Make a graph that shows the number of blue tiles in each square pool. Make a graph that shows the number of white tiles in each square pool.
- As the number of the pool increases, how does the number of white tiles change? How does the number of blue tiles change? How does this relationship show up in a table and in the graph?
- Use your graph to find the number of blue tiles in the seventh square.
- Can there ever be a border for a square pool with exactly twenty-five white tiles? Explain why or why not.

Next we can increase the demand of the problem so that students will look for patterns and make generalizations to help with predicting what will happen in the case of a very large pool. (See fig 1.)

- Find the number of blue (white) tiles in the 10th pool. The 25th pool. The 100th pool.

- If there are 144 blue squares, what is the side length of the square pool including the border? How many white tiles are needed for the border?

Looking into the Classroom for Grades 5–6

Some students continue to build the squares using tiles, and they notice relationships between the numbers of blue and border tiles as they build and then record their data in a table. Some students find that using grid paper is helpful. For some students, the act of building or drawing the pools suggests the relationship between the number of blue and the number of white tiles. The number of white tiles is four times the number of blue tiles on a side plus four for the four corners. Some students may use a table to find the number of blue or white tiles in the 10th pool. But some students begin to reason about the patterns. “In the 10th pool, the square formed by the blue tiles is a 10 × 10 square, so there are 100 blue tiles. There are 144 tiles, so 144 total tiles minus 100 blue tiles equals 44 white tiles.” Some students may first reason about the number of white tiles, whereas other students may draw the 10th pool and reason from the picture. As they continue to explore these problems, they begin to notice other patterns. Some notice that the number of white tiles will always be a multiple of 4. Some students question whether every multiple of 4 is a white-tile total (see fig. 6).

After discussing the patterns in the table, the teacher suggests that the class explore the graphs of these patterns. The graphs suggest that the relationship between the number of the square pool

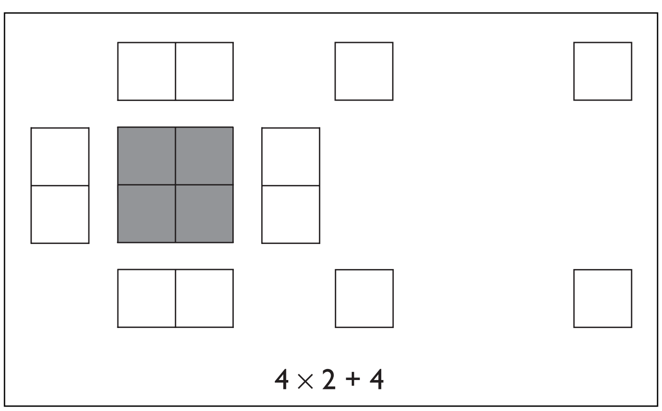

Fig. 6. The number of white tiles is always a multiple of 4.

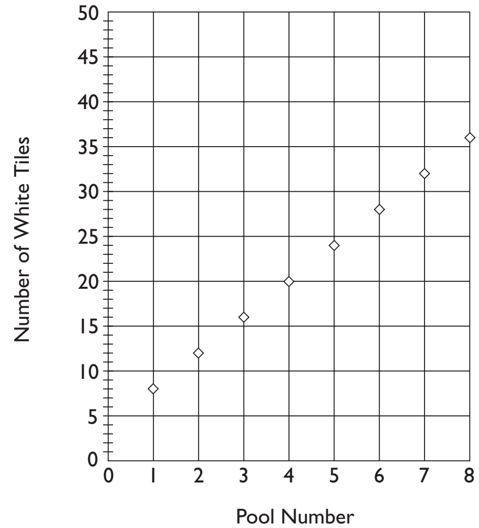

Fig. 7. A graph of the white tiles for each pool

and the number of white tiles can be represented by a straight line, and that the number of the square pool and the number of blue tiles lie on a curved line. In later grades the first pattern is called a *linear function,* and the latter pattern is called a *quadratic function*. The students can use the graph to find the number of white tiles (see fig. 7) or of blue tiles (see fig. 8) given the pool number, and, conversely, given the number of white tiles or blue tiles, they can find the pool number. By graphing the white- and blue-tile patterns on the same grid, students can use the graphs to reason about when the two patterns are equal or when one is greater than the other. (See fig. 9.)

Where Is the Algebra?

Throughout the grade-level examples of versions of the pool-design problem, children are being challenged to observe patterns in the growth of the numbers of blue tiles and white tiles needed to make the next square pool and to build connections between the physical representations and their verbal descriptions. This activity involves an informal interaction with variables. In the early grades, the focus is on the number of each kind of tile and which is more. Even here the problem has the potential to challenge each child at his or her level of interest and insight. All children can sort and count the blue and white tiles, but the teacher can ask questions that push beyond, for example:

> Build your own design out of white and blue tiles. How many blue tiles and white tiles does your design have?

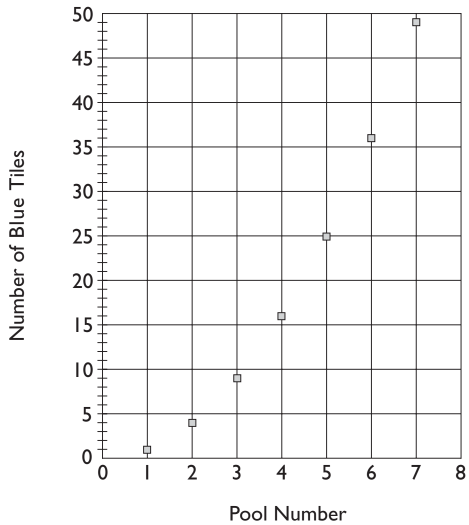

Fig. 8. A graph of the blue tiles for each pool

As the problem moves up the grades in elementary school, the questions asked push toward generalization. The children are challenged to find a way to describe the relationship between the number of blue tiles and the number of white tiles and the position in the sequence of pool designs. The ways to represent the change in the variables from on: pool to the next also become more varied over time. Verbal descriptions, tables, graphs, and symbolic expressions are all legitimate ways to express the relationships.

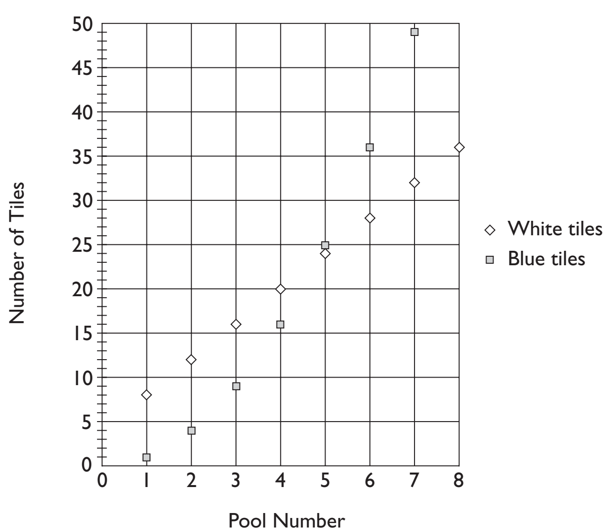

Fig. 9. Putting the graphs on the same coordinate axis helps to compare the two patterns.

Looking for Algebraic Reasoning

Many situations in elementary school mathematics can give teachers an opportunity to generalize and represent mathematical ideas and processes. In this article we offer a geometric setting that illustrates how mathematical ideas can be developed from the study of problems and how algebra emerges as a way to generalize and represent these ideas. Many other settings situated in number, data, and measurement are fruitful sites for developing algebraic reasoning. The following set of questions can serve to organize a classroom discussion in a variety of settings. The wording and choices of representations will vary depending on the experiences of the students.

- What are the variables in this situation? What quantities are changing?
- How are the variables related?
- As one variable increases, what happens to the other variable?
- How can you represent this relationship using words, concrete objects, pictures, tables, graphs, or symbols?
- How can you build connections among representations?
- How can you use this relationship to predict information about the variables?

Bibliography

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: The Council, 1989.

———. *Professional Standards for Teaching Mathematics.* Reston, Va.: The Council, 1991.

National Council of Teachers of Mathematics Algebra Working Group. *Algebra in the K–12 Curriculum: Dilemmas and Possibilities.* Final Report to the Board of Directors. East Lansing: Michigan State University, 1995.

Steen, Lynn Arthur. "The Science of Patterns." *Science* 240 (29 April 1988): 611–16.